Nonlinearity and Chaos in Macroeconomics
and Financial Markets

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Nonlinear modeling has exploded in recent years, aided by new econometric techniques and powerful computer programs. Notably, the 2003 Noble Prize went to two economists who gave the field new methods of analyzing time series data, Clive Granger and Robert Engle. Their contribution, cointegration and autoregressive conditional heteroskedasticity, ARCH, respectively, opened the door to the analysis of large sets of financial and macroeconomic data. These techniques revealed dynamics that were previously obscured by systematic noise, time varying volatility, and non-stationary trends in the data. Looking at the strange data pouring out of the economy with new insight forced the question, "What type of system could generate this?" The question inspired the half answer, half question: "A nonlinear system?" This paper is an introduction to the growing use of nonlinear dynamic system models in economics, their application in financial asset pricing, and most interestingly, how chaos theory could be the key to putting it all together.

A basic math course will introduce students to a linear equation like those describing straight lines. Dynamic systems, which are groups of linked equations, are considered "linear" if the global properties of the system can be completely identified by their local behavior (Pesaran 1992). Knowing the equation of a line, which is the system in this case, allows us to generate all possible points from any given point. If the motion of interest can be identified in a particular time frame by the system of equations, and those equations give accurate descriptions of any subsequent time frame, the system is linear. The most basic example is a swinging pendulum. Similarly, nonlinear equations are equations that do not follow a linear trend. Trigonometric functions and periodic functions are all nonlinear. Nonlinear equations are powerful tools in modeling any oscillating phenomenon, which makes them well suited for examining macroeconomic fluctuations and business cycle research.

Unlike linear systems, nonlinear dynamic systems do not have an obvious link between their local and global dynamics. Observing a system at one time may not lead to insight about their evolution. Nonlinear systems can be very complex and observations may not be enough to reveal the underlying governing equations. These systems can incorporate a nonlinear function that links a variable's current value any past values. It can also link to past values of other variables, including random shock variables. This gives nonlinear economic systems their history dependence. Since dependence and persistence are thought to be prevalent in the economy, building a system that can inherently behave in that manner is beneficial (Pesaran 1992). By adjusting the linkage equation, nonlinear systems can capture long memory behaviors and internal dynamics. Properties like long memories and endogenized dynamic and persistent shocks put nonlinear modeling in a position to improve business cycle models. This is the power of nonlinear systems. They allow researchers to model more realistic systems.

Traditionally business cycle research treats fluctuations as deviations from a steady state caused by exogenous "shocks" like fiscal and monetary policy changes, and changes in technology. These shocks and their propagation through the economy are evaluated using Dynamic Stochastic General Equilibrium (DSGE) models (Linton 2001). DSGE models are complex, and still leave shocks outside their framework. The economy as a whole is a closed system, yet current theory treats it in pieces. However, nonlinear systems can create and propagate fluctuations without any influence from outside the model. Incorporating nonlinear equations can produce both stable and stochastic-like responses without exogenous parameters. Each individual piece changes with the others, and allows economists pull all the pieces together and capture periodic behavior within a single model.

The most exciting feature of nonlinear systems is their ability to display chaotic dynamics. Unfortunately, defining chaos is trickier than explaining linear and nonlinear equations. First, mathematical models come in two varieties, deterministic or stochastic. Medio (1992) defines a deterministic system as "one containing no exogenous stochastic variables." An experiment with this type of system will yield exactly the same results when performed in exactly the same way. Conversely, a stochastic system will give different outcomes when repeated, and the different outcome states can only be described by a probabilistic function. A roulette wheel or rolling dice are stochastic systems. Second, both linear and nonlinear systems can be deterministic. Knowledge of all a system's equations and the links between them will lead to accurate prediction in either system. Linear and nonlinear systems can also be stochastic if they contain a random term within the system, i.e. the same conditions and process will yield a different outcome when it is repeated (Bau). Much economic data has this randomness, but it comes from agents and markets that are presumably rational and deterministic. Can we create a deterministic system to give random-like output without incorporating a pure stochastic element? Mathematicians have shown that surprisingly small systems of nonlinear deterministic equations can exhibit random, "chaotic" behavior. Nonlinear systems as small as three equations can lead to random-like results. This randomness within a deterministic system is chaos.

Chaos has several operating definitions. First and most mathematically, chaos is a bounded deterministic system with a positive Lyaponuv exponent. We will return to this definition when we explore the tests for chaos. A more intuitive definition came from the Royal Society of London in 1986, where chaos is 'stochastic behavior occurring in a deterministic system.' A chaotic system will show random results to a repeated experiment on such a deterministic system. This may be counterintuitive to the common sense proposed in the preceding paragraph: knowledge of a system's current state and evolutionary path should lead to predictions of all future states in the absence of random variables. A defining characteristic of chaotic systems is that they have sensitive dependence to initial conditions, SDIC. Any degree of uncertainty in the initial data, as often occurs in measuring, will grow as the system evolves. Moreover, the errors will propagate in unpredictable ways, making forecasting impossible. For example, a pendulum is only predictable to some degree of measurement error. In reality, even its motion pendulum can become unpredictable with imperceptible measurement errors. While any error can grow immensely when considering an unbounded system, errors can also grow exponentially within a nonlinear, bounded system (Bau). Therefore, a more complete definition of chaos would be a bounded, deterministic system with high sensitivity to initial conditions, shown by a positive Lyaponuv exponent.

All of this may seem to be unremarkable results to those outside mathematical theory. The trick to seeing chaos' application to economics is thinking about the result backwards. If simple deterministic systems can give random results, then the seemingly random data that economists often encounter might not be coming from a random system. The generating system could be deterministic and perhaps the economy can be explained by a relatively simple nonlinear system. Although economists are still quite far from this unifying equation, they have made strides toward it.

One route toward finding a nonlinear underlying system in the economy would be to show that the data itself demonstrates nonlinear or chaotic properties. This is the idea behind much economic-chaos literature. Researchers developed tests for chaos and nonlinearity in data. Showing the data is chaotic is more beneficial to research but also very difficult to prove. Showing nonlinearity is not as fruitful, and also more argument prone.

There are two major classes of tests for chaos within data. The first is ways to look at the paths or trajectories of the data when the system's initial conditions are adjusted slightly. Mathematicians do this by estimating a Lyaponuv exponent. The Lyaponuv exponent is a measure of the average divergence (or convergence) between experimental data, or "trajectories," generated by systems with infinitesimally small changes in their initial conditions (Dechart 1992). If the data deviates exponentially from each other when we have this very small tweak on a deterministic model, it will have a positive Lyaponuv exponent. If the paths converge back to a steady state, then the Lyaponuv exponent will be negative. A positive exponent signals that the system must have sensitive dependence to initial conditions and therefore is chaotic (Barnett Nonlinear). While this method is fruitful, it is often difficult to implement.

The second type of test for chaos examines the dimensionality of the system. It may seem easy to explain that a square has two dimensions, and a line has one, but it is significantly more complicated for chaotic systems since they have non-integer dimensionality. The "fractional" dimensionality is what coined the term "fractal" for shapes generated by chaotic data. If a system does not have an integer dimension, then it could be chaotic, but how do we define a non-integer dimension? We develop this idea geometrically.

Dimension is defined as the ratio of the logarithm of "self similar" pieces to the logarithm of the magnification factor needed to make these pieces identical to the original, "whole", piece. For instance, a line has dimension 1. Breaking a line into two identical pieces yields two self similar pieces. The numerator of the equation for dimension becomes log 2. Intuitively, it would take a magnification factor of 2 to "blow up" each piece to the same size as their parent line. This makes the denominator also log 2[1]. Hence the dimension of a line is:

$$\text{Dimension} = \frac{\log\left(\text{self similar pieces}\right)}{\log\left(\text{magnification factor}\right)} = \frac{\log 2}{\log 2} = 1$$

The analysis can be repeated for a square, cube, and any other common shape for whom we can intuitively name their dimension.

How does this geometric interpretation pertain to the world of economic data? Economists don't often analyze geometric shapes. Instead think of a graph of time series data, such as real GDP or a stock's percentage return. Could this data be arranged in a geometric way which would have the properties of being broken into self similar pieces that could be magnified to replicate the parent? While a daunting mental task, this is the physical interpretation of how data is tested for chaos, and dramatically shows the

[1] In general, a line can be broken into any number of identical pieces, N. But each piece N will still need to be magnified N times to match the parent. This makes the dimension = $\frac{\log N}{\log N} = 1$

complicated nature of the endeavor, and also why there are conflicting theories. If it could be done, we could obtain a fractal dimension for the data, and show that it is chaotic. Then it could be possible, if difficult, to define the deterministic nonlinear system that gave the rules to "assemble" the data in the way that gave us a fractal picture.

Dimensionality analysis becomes extremely complicated with realistic chaotic systems. Chaos can exist in many different fractal dimensions. Unfortunately, the analysis gets more and more complex the larger the fractal dimension being searched in, and the tests for chaos become weaker. Unless data displays low-dimensional chaos, it may be undetectable to current tests. This is a significant obstacle to chaotic-economic theory, and one of the main reasons the literature has not reached a consensus on the existence of chaotic dynamics in data.

Most testing for nonlinearity and chaos uses data from financial markets. To see why requires a basic asset pricing framework. The most widely accepted idea about the nature of financial markets is Efficient Market Hypothesis[2]. The Efficient Market Hypothesis states that at any given time asset prices reflect all available information. The hypothesis stems from implications of the martingale and random walk models, where the next day's valuation is expected to be the same as todays. The model implies that any changes in value are unpredictable, given all the information available on any given day.

Following Barnett and Medio, we can express a stochastic variable, x, following a martingale if $E_t\left[x_{t+1} \mid \Omega_t\right] = x_t$, where Ω is all information available at time t. This equation shows that the best estimate of next period's value of x, is x itself. The information contained in Ω is completely reflected in x, the price itself. Intuitively, the

[2] The Efficient Market Hypothesis is credited to Eugene Fama.

information is used up, so it cannot be helpful in predicting rates of return in the next period. (Barnett 1998). The random walk hypothesis, a similar theory, is more restrictive than the martingale model. The martingale model only requires independence of available information, Ω in the above example, and the expectation of price changes. Random walks require independence of higher conditional moments such as the variance. The independence of higher moments gives random walk theory the attribute that price movements will not follow any trends. Similar to the martingale model, past price movements cannot be used to predict future price movements either. In both specifications, the independence of today's information and tomorrow's prices implies efficient markets.

It is important to note that efficient markets imply there are no exploitable profit opportunities (Barnett Nonlinear). If this is true then trading on the stock market is a game of chance and not of any skill, but traders buy assets they think are undervalued at the hope of selling them at their true price for a profit. If market prices already reflect all information available, then where does the trader draw this privileged information from? Since there are thousands of very well informed, well educated asset traders, backed by many data researchers, buying and selling securities quickly, logically asset markets should be very efficient and profit opportunities should be minimal. If the Efficient Market Hypothesis holds, then we should see profits in asset markets exhibiting random properties, and we can test the randomness for chaos and nonlinearity.

If there is nonlinearity or chaos, then the exciting possibility of forecasting asset prices exists. However, if we can predict next period's prices, then it must not be independent of the current information set, and last period's price was not the best estimate. Predictability will reject the Efficient Market Hypothesis (Pesaran), which is how we came to test for chaos originally. However, if we turn away from the martingale and random walk models we find a growing body of literature on nonlinear financial frameworks.

Nonlinear models constitute a new structure for asset pricing models, but are these departures justified? Experimental evidence shows that investors risk aversion and expected returns are nonlinear, as well as the terms of many option contracts (Barnett Nonlinear). Nonlinear foundations retain efficient markets without forcing the assumptions of martingale differences and random walks on the model, and still generate the correct empirical output. If we know this underlying system, it is possible to obtain short run, profitable, trading rules[3] (Barnett 1998). This is a very exciting possibility, and the first step toward it is to find evidence for chaos.

There are several popular tests for nonlinearity and chaos in economic time series data. The first is the correlation dimension test for the presence of chaos, developed by Peter Grassberger and Itamar Procaccia. Instead of returning a quantitative answer, like the actual dimension, the test returns a probability of what the dimension might be[4] (Barnett Nonliner). As mentioned, there are daunting mathematical hurdles to directly obtaining the fractal dimension of time series data, especially as the dimensionality increases. This test simplifies the calculations and leads to fewer non-convergent solutions. Unfortunately, it also loses power as the size of data sets decreases in addition to its weakness at detecting high dimension chaos. To address this, the BDS test for

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[3] We can not develop long term accurate forecasts though because of the chaotic system's sensitive dependence on initial conditions.

[4] A more detailed discussion of these tests would only be possible with a more mathematical representation of chaos, which is outside the scope of this paper. For more detailed derivations see Barnett, Medio, and Serlitis.

white noise in errors was developed by William Brock, Davis Dechert, Blake Lebaron, and Jose Scheinkman, and is less sensitive to the size of the data set. This test's major shortcoming is that it does not test directly for nonlinearity or chaos. It can only reject a null hypothesis of white noise, or uncorrelated errors, which gives indirect evidence of nonlinearity in the data. In contrast, the NEGM test does look directly for a chaotic system by appealing to the hallmark of chaotic systems: a positive Lyaponuv exponent. Unfortunately it can a difficult test to implement. Other tests include the White Test and the Kaplan Test, each with their own benefits and drawbacks.

Proving or disproving the existence of chaos in macroeconomic data would be a significant discovery, and controversy and arguments follows anyone who claims either. There is strong support in economics for both the significance of linear models, and the advantages of nonlinear models. The obvious question is, if nonlinear models clearly outperform linear models, then why is the field flooded with linear pedagogy? Perasan and Potter emphasize that the prevalence of linear models in economics has nothing to do with any theory that the economy is inherently linear, only that linear models are easier to work with. Linearity is a strong assumption, and should be easier to reject than a more feasible null hypothesis. One only has to look at the curvature of utility and production functions or the periodicity of output and GDP to see the prevalence of nonlinearity. Also, the economy is subject to external disturbances from the nonlinear environment, like the weather. Therefore, a test should reject linearity just on this basis, but it is not the nonlinearity we are searching for.

Unfortunately, Granger and other prominent economists within the field have shown that there is not much empirical evidence for nonlinearity in macroeconomic literature (Franses 2001). However, this isn't a refutation of the nonlinear field. Rather, it illustrates several of the biggest problems with the research. The largest problem in testing macroeconomic data for chaos and nonlinear dynamics is simply the quality of the data. Most of the tests currently available are strongest with long strings of data measured at small intervals. Unfortunately this is the most uncommon type of data available to economists. To make up for this, analysts must either sacrifice short interval measurements for larger data sets, or take very small data sets with short intervals. Additionally, long data sets have the problem of being non-stationary. These problems leave holes in most empirical results that hold back a consensus.

These problems are most prevalent in aggregate macroeconomic data. Granger (2001) shows that aggregation of independent series of data makes it much harder to detect underlying nonlinear dynamics. The aggregation in effect hides the nonlinear or chaotic signals the tests look for. Current tests used to detect nonlinear structure often fail to find evidence of nonlinearity in aggregated data, even if that data was actually generated by a nonlinear process. Also, applying these tests to time series data does not distinguish between nonlinearity or chaos in the economic behavior and deeper nonlinear chaotic processes, like weather patterns, in the data generating process (Barnett Unsolved). Hence, a positive result would not be a definitive finding of chaos. Finally, there are significant losses of nonlinear "signals" when testing data over wider time intervals. Nonlinear data collected weekly but tested for nonlinearity on a yearly basis is likely to fail the tests. These facts illustrate that modeling with nonlinear systems is most accurate dealing in short time frames, with data that are direct outcomes of market processes, such as interest rates and asset prices, as opposed to aggregates like the money supply that are less insulated from other, possibly nonlinear, processes (Granger 2001). This fact greatly expanded the literature in the field of financial data modeling.

The literature surrounding testing for chaos remains the most ambiguous. Shintani and Linton (2001) reject the hypothesis of a positive Lyaponuv exponent in international real output time series, while Barnett and Chen (1988) find chaotic evidence in weekly Divisia monetary aggregates using the correlation dimension test[5]. As often happens in this field of research, a work by Ramsey, Sayers and Rothman (1990) raised questions about Barnett and Chen's positive result, finding no evidence using the same data and the same test, and also negated Scheinkman and LeBaron's (1989) positive chaos result. This pattern is repeated for many published results using different macroeconomic data sets and tests for chaos.

There is more empirical support in the literature when testing strictly for nonlinearity. A positive result for nonlinearity could reasonably imply chaos, though not with certainty. Scheinkman and Lebaron's previous result shows strong evidence of nonlinearity, if its chaotic results are less apparent. Barnett (1998) reports similarly strong backing for nonlinearity in financial data published by Frank and Stengos (1989), and Abhyankar, Copeland, and Wong (1995, 1997). These results came from using the BDS test on U.S. weekly returns, the correlation dimension test on gold and silver returns, and the BDS and NEGM test on some of the major world stock indices. While these results showed little or no evidence of low dimensional chaotic dynamics, there still may be high dimension chaos. It is possible that the underlying nonlinear structure to the economy is so complex that the chaotic dynamics it exhibits are of a higher dimensionality than current techniques can distinguish.

[5] See Dechart and Gencay 1992 for a description of testing for Lyaponuv exponents.

To sum up the state of the field: there is no reason to suppose that the economy is linear, so we should be able to find evidence of nonlinearity in economic data. But there has been no proof of nonlinearity using the available tests, so we come back to assuming linearity. It seems the only consensus is that there is no consensus.

If econometricians could shore up the holes, and prove the existence of chaos in financial data, what would be gained? First, nonlinear models can outperform traditional models, especially in financial markets. Following from this, we examined how nonlinear models may necessitate using chaos theory to work backward through the stochastic data. Proving that the data is chaotic would prove there is a deterministic underlying system generating the data. The proof would be a giant step toward clarifying the "nature" of the economy. Currently models are forced to treat shocks, the driving force of economic fluctuations and the reason for the unpredictability of future conditions, as exogenous and completely random. Chaotic nonlinear systems can endogenize shocks. If the economy is chaotic, then we can create a complete and closed model. This development would aid significantly in short run forecasting.

Unfortunately until better tests are developed we may be forced to argue the existence of chaos and a nonlinear underlying economic structure on philosophical grounds. Even with a consensus there would still be the task of assembling the dynamic system at the heart of the economy. However daunting the task, the grand implications of predictable financial markets, endogenized shocks, and clarification of the structure of the macroeconomy, give fruit to the labor. With these incentives the research and literature in this area is certain to grow in both size and importance as a field within economics.

Works Cited

Barnett, William A., Apostolos Serletis. 1998. *Martingales, Nonlinearity, and Chaos.* Department of Economics, Washington University-St. Louis Working Papers. URL: http://econwpa.wustl.edu:80/eps/em/papers/9805/9805003.pdf. 12/03.

Barnett, William A., Alfredo Medio, Apostolos Serletis. *Nonlinear and Complex Dynamics in Economics.* Department of Economics, Washington University-St. Louis Working Papers Working Paper. URL: http://econwpa.wustl.edu:80/eps/em/papers/9709/9709001.pdf. 12/03.

Barnett, William A., Melvin J. Hinich. "Has Chaos Been Discovered with Economic Data?." Nonlinear Dynamics and Evolutionary Economics. Ed. Richard H. Day, Ping Chen. Oxford: Oxford, 1993. 254-265.

Barnett, William, Yijun He. "Unsolved Econometric Problems in Nonlinearity, Chaos, and Bifurcation." Department of Economics, Washington University-St. Louis Working Papers Working Paper Working Paper. URL: http://econwpa.wustl.edu:80/eps/mac/papers/0004/0004021.pdf. 12/03.

Bau, Haim H., Yochanan Shachmurove. *Chaos Theory and its Application.* University of Pennsylvania: Center for Analytical Research in Economics and the Social Sciences Working Paper. URL: http://www.econ.upenn.edu/Centers/CARESS/02-02.pdf. 12/03.

Chen, Ping. "Searching for Economic Chaos: A Challenge to Econometric Practice and Nonlinear Tests." Nonlinear Dynamics and Evolutionary Economics. Ed. Richard H. Day, Ping Chen. Oxford: Oxford, 1993. 217-253

Dechert, W.D., R. Gencay. "Lyapunov Exponents as Nonparametric Diagnostic for Stability Analysis." Journal of Applied Econometrics. Vol 7 Supplement. (Dec 1992): S41-S7.

De Waegenaere, Anja, Robert Kast, Andre Lapied. 1997. *Non-Linear Asset Valuation on Markets with Friction*. Tilburg University Center for Economic Research Working Paper. URL: http://greywww.kub.nl:2080/greyfiles/center/1996/doc/112.pdf. 12/03.

Franses, Philip Hans, Timo Terasvirta. "Introduction to the Special Issue: Nonlinear Modeling of Multivariate Macroeconomic Relations." Macroeconomic Dynamics. Vol 2 Num 4. (1998): 461-465.

Granger, Clive W.J. "Overview of Nonlinear Macroeconometric Empirical Models." Macroeconomic Dynamics. Vol 5. Num 4. (2001): 466-481.

Hiemstra, Craig., Charles Kramer. "Nonlinearity and Endogeneity in Macro-Asset Pricing." Studies in Nonlinear Dynamics and Econometrics. Vol 2 Num 3 (Oct., 1997): 61-76.

Lorenz, Hans-Walter. Nonlinear Dynamical Economics and Chaotic Motion. New York: Springer-Verlag, 1989.

Medio, Alfredo. Chaotic Dynamics: Theory and Application to Economics. Great Britian: Cambridge, 1992.

Persaran, Hashem M., Simon M. Potter. "Nonlinear Dynamics and Econometrics: An Introduction." Journal of Applied Econometrics. Vol 7 Supplement. (Dec 1992): S1-S7.

Peters, Edgar E. Chaos and Order in the Capital Markets. New York: John Wiley & Sons, 1996.

Peters, Edgar E. Fractal Market Analysis: Applying Chaos Theory to Investment Economics. New York: John Wiley & Sons, 1994.

Potter, Simon M. "Nonlinear Time Series Modeling: An Introduction." Surveys in Economics Dynamics. Ed. Donald A.R.George, Les Oxley, Simon Potter. Great Britian: Blackwell, 2000. 1-24.

Rosser Jr., J. Barkely. From Catastrophe to Chaos: A General Theory of Economic Discontinuities. Boston: Kluwer, 2000.

Rostow, W.W. "Nonlinear Dynamics and Economics: A Historian's Perspective." Nonlinear Dynamics and Evolutionary Economics. Ed. Richard H. Day, Ping Chen. Oxford: Oxford, 1993. 14-17.

Scheinkman, Jose A., Blake LeBaron. "Nonlinear Dynamics and Stock Returns." The Journal of Business. Vol. 62, No. 3. (Jul., 1989): 311-337.

Shintani, Mototsugu, Oliver Linton. 2001. *Is There Chaos in the World Economy? A Nonparametric Test Using Consistent Standard Errors.* Vanderbilt University Department of Economics Working Paper.
URL:
http://www.vanderbilt.edu/Econ/wparchive/workpaper/vu01-w11.pdf